Higher Jacobi Identities

Alekseev, Ilia

Any Lie algebra by definition satisfies the following identities $[x_1,x_2]+[x_2,x_1]=0$ and $[x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]=0$, where $[a_1,...,a_n]$ denotes the left-normed Lie bracket. Moreover, it is easy to check that any Lie algebra satisfies one more identity: $[x_1,x_2,x_3,x_4]+[x_2,x_1,x_4,x_3]+[x_4,x_3,x_2,x_1]+[x_3,x_4,x_1,x_2]=0$. This motivates the following definition. A subset T of the symmetric group S_n is said to be Jacoby if any Lie algebra satisfies the similar identity where the sum is taken over all permutations in T. Similarly, a subset T is said to be 2-Jacoby if the identity is satisfied in any Lie algebra over a field of characteristic 2. The work is devoted to an investigation of Jacoby and 2-Jacoby subsets. A big family of Jacoby subsets $T_{k,l,n}$ of S_n was constructed. Moreover, a ‘vector space of all relations between left-normed brackets’ was constructed and a basis of the space induced by the subsets $T_{k,1,n}$ was found. It was proved that the class of 2-Jacoby subsets is closed under some operations. In particular, in contrast to the class of usual Jacoby subsets, the class of 2-Jacoby subsets is closed under symmetric difference, which makes it a Z/2-vector space. Moreover, it was proved that any 2-Jacoby subset can be obtained as a symmetric difference of several Jacoby subsets.