"Equal Powers Turn Out" - Conics, Quadrics, and Beyond

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Our project is motivated by the following property: “For four focal radii originating equiangularly from a focus of an ellipse, the sum of their squared reciprocals is a constant.” We generalize this property to different settings, adding segments, arbitrary powers, arbitrary origin, all conics and quadrics. Numerous unexpected sets of invariants are found with an application to a number theory problem. Our strategy is as follows: by experimenting on GeoGebra, we observe many invariants and propose various conjectures. To prove the conjectures about the conics, we use polar coordinates and transform the conjectures into identities of trigonometric series. The results are proved with the help of a power reducing formula and a weighted identity of arithmetic progression of cosines which we developed. Major results are as follows. First, we prove that the sums of the m-th power reciprocal of n equiangular focal radii originating from a focus of an ellipse are invariant for all $m<n$, independent of the orientations of the radii. For segments originating from an arbitrary point, the invariants hold only for even m. Second, we generalize the above invariant properties to all conics. Third, we generalize further to quadrics. Here, the equiangular segments are generated by the vertices of a regular polytope. The valid powers for the invariants of these cases can be completely determined. The situation, however, is more subtle. The proofs are done by reducing the problems to a key base, which specializes to a 3D version of the conics previously solved. As an application, our invariants allow constructively a family of infinite solutions to the real number version of the Prouhet-Tarry-Escott problem, which asks for two sets of integers with their first k power sums being equal.

Awards Won:

American Mathematical Society: Certificate of Honorable Mention