Pythagorean Quintuples and Quaternions

Ehrenborg, Theodore (School: Henry Clay High School)

This project, which is in number theory, explores the connections between quaternions and primitive Pythagorean quintuples. It is known that the square of a Gaussian integer (a complex number) is a Pythagorean triple a^2 + b^2 = c^2. Less is known about the relationship between quaternions, an extension of complex numbers, and Pythagorean quintuples a^2 + b^2 + c^2 + d^2 = e^2. We show that squaring a quaternion produces a subfamily of Pythagorean quintuples. Motivated by Conway and Smith's unique factorization theorem for the Hurwitz integers, we present a more general version of squaring a quaternion which generates a larger subfamily of Pythagorean quintuples. Using a counting argument and Jacobi's Four Square Theorem, we analyze the results based upon Hurwitz integers. Finally, we use a geometric approach to completely characterize all Pythagorean quintuples. We notice a similarity between the geometric approach and the squaring a quaternion approach in that they differ by a geometrically defined constant.

Awards Won:

Third Award of $1,000