

Optimal Bounds for a Gaussian Arithmetic-Geometric Type Mean by Quadratic and Contraharmonic Means

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In this paper, we present the best possible parameters α_i, β_i ($i=1,2,3$) and $\alpha_4, \beta_4 \in (1/2, 1)$ such that the double inequalities
$$\begin{aligned} \alpha_1 Q(a,b) + (1-\alpha_1)C(a,b) &< AG_{\{Q,C\}}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)C(a,b), \\ Q^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) &< AG_{\{Q,C\}}(a,b) < Q^{\beta_2}(a,b)C^{1-\beta_2}(a,b), \\ \frac{Q(a,b)C(a,b)}{\alpha_3 Q(a,b) + (1-\alpha_3)C(a,b)} &< AG_{\{Q,C\}}(a,b) < \frac{Q(a,b)C(a,b)}{\beta_3 Q(a,b) + (1-\beta_3)C(a,b)}, \\ C\left(\sqrt{\alpha_4 a^2 + (1-\alpha_4)b^2}, \sqrt{(1-\alpha_4)a^2 + \alpha_4 b^2}\right) &< AG_{\{Q,C\}}(a,b) < C\left(\sqrt{\beta_4 a^2 + (1-\beta_4)b^2}, \sqrt{(1-\beta_4)a^2 + \beta_4 b^2}\right) \end{aligned}$$
 hold for all $a, b > 0$ with $a \neq b$, where $Q(a,b)$, $C(a,b)$ and $AG(a,b)$ are the quadratic, contraharmonic and Arithmetic-Geometric means, and $AG_{\{Q,C\}}(a,b) = AG[Q(a,b), C(a,b)]$. As consequences, we present new bounds for the complete elliptic integral of the first kind. Keywords: Arithmetic-Geometric mean, Complete elliptic integral, Quadratic mean, Contraharmonic mean