Optimal Bounds for a Gaussian Arithmetic-Geometric Type Mean by Quadratic and Contraharmonic Means

Shen, Junxuan (School: Hangzhou Foreign Languages School)

In this paper, we present the best possible parameters $\alpha_i, \beta_i\ (i=1,2,3)$ and $\alpha_4,\beta_4\in(1/2,1)$ such that the double inequalities

$$\begin{align*} \alpha_1Q(a,b)+(1-\alpha_1)C(a,b)&<AG_{Q,C}(a,b)<\beta_1Q(a,b)+(1-\beta_1)C(a,b),\\ \qquad\ Q^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b)&<AG_{Q,C}(a,b)<Q^{\beta_2}(a,b)C^{1-\beta_2}(a,b),\\ \frac{Q(a,b)C(a,b)}{\alpha_3Q(a,b)+(1-\alpha_3)C(a,b)}&<AG_{Q,C}(a,b)<\frac{Q(a,b)C(a,b)}{\beta_3Q(a,b)+(1-\beta_3)C(a,b)},\\ C\left(\sqrt{\alpha_4a^2+(1-\alpha_4)b^2},\sqrt{(1-\alpha_4)a^2+\alpha_4b^2}\right)&<AG_{Q,C}(a,b)<C\left(\sqrt{\beta_4a^2+(1-\beta_4)b^2},\sqrt{(1-\beta_4)a^2+\beta_4b^2}\right) \end{align*}$$

hold for all $a, b>0$ with $a\neq b$, where $Q(a,b)$, $C(a,b)$ and $AG(a,b)$ are the quadratic, contraharmonic and Arithmetic-Geometric means, and $AG_{Q,C}(a,b)=AG[Q(a,b),C(a,b)]$. As consequences, we present new bounds for the complete elliptic integral of the first kind. Keywords: Arithmetic-Geometric mean, Complete elliptic integral, Quadratic mean, Contraharmonic mean