Classifying Quaternion Identities

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This number theory project investigates identities found by multiplying together quaternions in L[x,y,z,w], the Lipschitz quaternions L adjoined with the indeterminates x, y, z, w. Recall that quaternions are 4-dimensional complex numbers. These identities provide solutions to equations where a sum of p squares in m variables is equal to an nth power. We present a rigorous definition that captures the intuitive notion of when two such identities are equivalent. This definition implies that the true structure of this problem involves the group action of the direct product of two signed symmetric groups. Using two complementary methods, we compute the number of equivalence classes for n = 1, 2, 3, 4, where n is the number of quaternion factors. We move to the case concerning products of complex numbers, namely Z[i][x,y]. Using the fact that the Gaussian integers are commutative under multiplication, we characterize these equivalence classes, thus also providing an enumeration.