On a Generalization of Artin's Conjecture for Primitive Roots in Gaussian Integers

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We propose a generalization of Artin's conjecture on primitive roots to the ring Z[i] of Gaussian integers. We conjecture that for a fixed positive integer q, every non-zero Gaussian integer a that is not +1 or -1, generates a cyclic subgroup of the multiplicative group of Z[i]/p of residue index q for infinitely many prime ideals p. In several special cases we reduce it either to the classical Artin's conjecture, or to its extension for near-primitive roots, the Golomb's conjecture. We divide the conjecture into three cases: when a is on the real axis, when a is on the imaginary axis, and when a is on neither axes. We conclude by showing that for every a, we have that the sum of d(a,q) over all positive q equals 1, where d(a,q) is density of the prime ideals p yielding subgroups of index precisely q.