Ramanujan Congruences for Tangent Numbers

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Euler numbers E_n appear in many places of classic analysis, combinatorics, and topology. They can be defined as Taylor coefficients of the function tanh x + sech x or (in the terminology of Arnold) as the number of A_n-snakes. They can be defined also as numbers of alternating (or zigzag) permutations. For example, {1324, 1423, 2314, 2413, 3412} is a list of alternating permutations on the set {1, 2, 3, 4}, and, $E_4=5$. The numbers E_{2n} with even indices are called secant numbers or zig numbers. Similarly, the numbers E_{2n+1} with odd indices are called tangent numbers or zag numbers. Euler numbers were studied by many famous scientists as Euler, Ramanujan, Carlitz, Arnold, etc. Ramanujan has established the following congruences for secant numbers: $E_{4n} \equiv 5 \pmod{60}$, $E_{4n+2} \equiv -1 \pmod{60}$. We extended this result for Euler tangent numbers. We proved that $E_{4n+1} \equiv 16 \pmod{720}$, $E_{4n+3} \equiv -272 \pmod{720}$. We also proved, that the number 720 here can not be improved. Tangent numbers are closely related to Bernoulli numbers B_n and Genocchi numbers G_n. For example, applications of our results for Genocchi numbers gives us the following congruences modulo 45: $G_{12 n} \equiv 3 n$, $G_{12 n + 2} \equiv -6 n -16$, $G_{12 n + 4} \equiv 3 n +1$, $G_{12 n + 6} \equiv -6 n - 3$, $G_{12 n + 8} \equiv 3 n +17$, $G_{12 n +10} \equiv -6 n + 25$.