Farey Sequences and Ford Spheres in Higher Dimensions

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Farey sequence F_n of order n is the sequence of all irreducible fractions between 0 and 1 with denominators no greater than n arranged in increasing order. Farey sequence has a geometric interpretation that it can be represented by a sequence of 2D circles, known as Ford circles, in which consecutive circles are tangent to each other and each term in the Farey sequence corresponds to the x-coordinate of a Ford circle. Algebraically, terms of Farey sequence can be computed rapidly by “non-conventional addition of fractions”, defined by adding up the numerators and denominators respectively. In this project, we extend the correspondence between Farey sequence and Ford circles to higher dimensional spaces. By lifting all 2D Ford circles to 3D spheres and introducing more 3D spheres to keep the tangency property, we are able to define “3D Farey sequence” that corresponds to the aforementioned 3D tangent balls, and at the same time satisfy a similar non-conventional fraction-addition rule. With the same line of approach we can generate “4D Ford spheres” that keep the tangency property and define the corresponding “4D Farey sequence” with the non-conventional fraction-addition rule. Our result is novel and the construction is non-trivial. Although the extension into an even higher dimensional space greater than 4 is difficult and still missing, we believe that our project sheds new light on the study of Farey sequence and Ford circles.

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