Counting Visible Points on Square Lattice by Arithmetic Functions With Asymptotic Behavior

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For a fixed positive integer $b \in N$, let the b-sight lines be defined by $f(x)=ax^b$, for $a \in Q$, with the origin O as the observing point (the position of the eyes). A point in the square lattice $V(m)=\{ (i, j) \mid i,j \in N, 1 \le i \le m, 1 \le j \le m\}$ is said to be b-visible if it is the “first” point in $V(m)$ that can be seen from the origin O through any sight line of the form $f(x)=ax^b$, for some $a \in Q$. Let $H_b (m)$ denote the total number of b-visible points in $V(m)$. Our goal in this project is to enumerate $H_b (m)$ and we show that it can be expressed by Möbius function. When b=1, due to symmetry, $H_1 (m)$ can be further reduced to a very neat formula in terms of Euler function. Moreover, by a probability result in literature, we obtain a non-trivial asymptotic limit $\lim_{m\to\infty} H_b (m)/^2 =1/\zeta(b+1)$ where $\zeta(s)$ is the Riemann-Zeta function. Finally, assuming that we now observe lattice points in $V(m)$ from another square $S(k)=\{(r,t) \mid 0 \le r \le k, 0 \le t \le k\}$, not limited to just the origin O. To see every lattice in $V(m)$, we show that it can be done from $S(k)$ whose side length k is no more than $A \cdot \sqrt{\pi(m)}$, where $\pi(m)$ is the number of primes less than or equal to m, and $A=3/\sqrt{1- 8/9 \ln(2.5)} \approx 6.965$. Our result is novel and interesting as it links counting in combinatorics with arithmetic functions in number theory and asymptotic behavior from analysis.