Nested Eggs: Where Brianchon, Pascal and Poncelet Meet

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Brianchon's and Pascal's theorems are two famous theorems in geometry that are known to be dual to each other. Despite their fundamental status, they are of much academic interest and have been used to derive surprisingly deep and far-reaching geometrical implications. This study explores if new properties can be found when the two theorems are applied to bicentric hexagons, which are hexagons that are both cyclic and tangential. This study started by constructing two families of hexagons. Firstly, we extended every other sides of a bicentric hexagon into the form of a hexagram followed by connecting its six vertices into a new hexagon. This procedure could be repeated and was denoted as operation S. Secondly, we drew tangents to the vertices of the circumcircle of a bicentric hexagon. The intersections of adjacent pairs of tangents will form a new hexagon. This procedure could be repeated and was denoted as operation T. By applying the properties of collinearity, concurrency and the principle of polarity, several interesting properties of bicentric hexagons were identified. In particular, it was found that all bicentric hexagons constructed under operations S and T are Poncelet hexagons (i.e. hexagons inscribed in a conic and simultaneously circumscribed by another conic). Furthermore, we studied new hexagons that were constructed under a composite of the S and T operations. We found that if the number of S and T operations were fixed respectively, the hexagons thus produced would be the same regardless of the order of implementation of the two procedures. That is, the two operations are commutative.

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