Patterns in the Continued Fraction Expansion of Various Infinite Series

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Continued fractions (CFs) provide well established advantages over other numeric representation systems, including greater accuracy, elimination of round-off and truncation errors, and a variety of fundamental symmetries. CF's have drawbacks in application, however, due to the arduousness of converting them to and from other numeric representations. For example, conversions between various infinite summations and their CF counterparts are often utilized in algorithms underlying modern cryptosystems. Yet traditional conversion methods remain slow and protracted, thus thwarting these cryptosystems' goals of speed and efficiency. This research uncovers a new, more efficient method of infinite sum/CF conversion. We first consider the CF expansion of $\sum\limits_{i=0}^\infty\frac{1}{T_{\ell^i}(x)}$ where T_ℓ is the ℓ^{th} Chebyshev Polynomial of the first kind, expanding on the works of Cohn [1] and Shallit [2]. Upon gaining insights into the structural and symmetrical properties of these CF expansions -- including previously undiscovered embedded patterns -- we develop a formula that computes the CF expansion of $\sum\limits_{i=0}^\infty\frac{1}{T_{\ell^i}(x)}$, solving the odd Chebyshev problem posed by Cohn in [1]. We then generalize and apply these insights and patterns to a broader class of polynomials and introduce a novel, more encompassing closed formula that computes the CF expansion of $\sum\limits_{i=0}^\infty\frac{1}{f^i(x)}$ for any $f(x)\in\mathbb{Z}[x]$ such that $f(x)\equiv kx \bmod x^2$ for some $k\in\mathbb{Z}$, thus expanding the bounds of our initial conjecture. In an effort to prove our results, we further generalize by applying our findings to the sum $\sum\limits_{i=0}^\infty\frac{a_i}{f^i(x)}$ where $a_i\in\mathbb{Z}$ for $0\leqslant i \leqslant n$.

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