Geodesics in the Discrete Heisenberg Group

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A wonderful fact called Thurston's geometrization conjecture states that there are essentially only eight three-dimensional geometries. One of them is Nil geometry, which is a Lie group called the Discrete Heisenberg group. This group is homeomorphic to usual Euclidean space, so in order to feel its metric structure one has to investigate its geodesic curves, which are arcs of helices. Large scale properties of this geometry are closely related to geometric properties of its two generator subgroup called the Discrete Heisenberg group. There is a problem of describing geodesic words in this group. In our research, we present a complete solution. More precisely, we give the solution in terms of closed polygonal chains on a plane and describe a beautiful connection between geodesic words and minimal perimeter polyominoes. Thus, we give a complete classification of the language of geodesics in the Discrete Heisenberg group. It is remarkable that this group is a free nilpotent group of rank two, so our results limit the space of geodesics in any nilpotent group of rank two. Our methods also give a new approach to the rationality of the geodesic growth of groups problem.

Awards Won:
American Mathematical Society: First Award of $2,000