Counting Visible Points on Square Lattice by Arithmetic Functions With Asymptotic Behavior

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For a fixed positive integer $b\epsilon N$, let the b-sight lines be defined by $f(x)=ax^b$, for $a\epsilon Q$, with the origin O as the observing point (the position of the eyes). A point in the square lattice $V(m)=\{ (i, j) \mid i,j\epsilon N, 1\le i\le m, 1\le j\le m\}$ is said to be b-visible if it is the “first” point in V(m) that can be seen from the origin O through any sight line of the form $f(x)=ax^b$, for some $a\epsilon Q$. Let $H_b (m)$ denote the total number of b-visible points in V(m). Our goal in this project is to enumerate $H_b (m)$ and we show that it can be expressed by Möbius function. When b=1, due to symmetry, $H_1 (m)$ can be further reduced to a very neat formula in terms of Euler function. Moreover, by a probability result in literature, we obtain a non-trivial asymptotic limit $\lim_{m\to\infty}H_b (m)/^2 =1/\zeta(b+1)$ where $\zeta(s)$ is the Riemann-Zeta function. Finally, assuming that we now observe lattice points in V(m) from another square $S(k)=\{(r,t) \mid 0\le r\le k,0\le t\le k\}$, not limited to just the origin O. To see every lattice in V(m), we show that it can be done from S(k) whose side length k is no more than $A\cdot\sqrt{\pi(m)}$, where $\pi(m)$ is the number of primes less than or equal to m, and $A=3/\sqrt{1- 8/9 \ln(2.5)}\approx 6.965$. Our result is novel and interesting as it links counting in combinatorics with arithmetic functions in number theory and asymptotic behavior from analysis.