Evaluation of Difficult Integrals Through an Equation Involving Fractional Derivatives

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The concept of fractional order derivatives, where the order s is not constrained to nonnegative integers, has long been a subject of significant interest. Various definitions, such as the Riemann-Liouville and Caputo derivatives, have been extensively studied, and it is particularly noteworthy that these definitions do not always coincide for the same value of s. In this work, a novel relationship is established, connecting a specific integral involving a function f to the limit of its fractional derivative as s approaches zero. Remarkably, this relationship holds for both the Caputo and Riemann-Liouville definitions under suitable conditions on f, despite their general lack of agreement. Beyond its theoretical elegance, this formula serves as a powerful tool for evaluating integrals. Its utility is demonstrated through applications to variations of Frullani's integral, the Dirichlet integral, and integrals related to the Euler-Mascheroni and Catalan constants, showcasing how these can be efficiently computed using this equality. Furthermore, the relationship is extended to the Fourier derivative, and a related equation involving a fractional integration operator is introduced. It is shown that the equation remains valid for the Fourier fractional integral, as well as for the Riemann-Liouville and Caputo integrals under appropriate conditions. Additionally, a proof is provided for the Fourier fractional derivative in the special case where f is an even periodic function with a mild constraint on its Fourier coefficients.

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