Prime Factorization of Linear Combination of Factorials

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There are many unsolved diophantine equations involving factorials in the field of number theory, such as Brocard's Problem. This research enhances the understandings of factorials with novel approaches. More specifically, we analyze the prime factorization of linear combination of large enough factorials, and proved the hypothesis that it can never be a power of an integer. We proved the following theorem. For any given positive integer n and integers a_0, a_1, ..., a_n there exists a positive integer M that for any integers m_0, m_1, ..., m_n larger than M, there exists a prime p that the exponent of p in the prime factorization of a_0m_0! + a_1m_1! + ... + a_nm_n! is exactly 1. We use induction on n with a stronger proposition including the range of such p to prove the main theorem. To prove the inductive hypothesis, we proceed with proof by contradiction assuming that for any prime factor, the exponent is at least 2. Here, we use prime number theorem and known bounds of gap of primes to bound product of primes in the interval and arrive to a contradiction. This study proved that the linear combination of factorials can never be a power of an integer when the factorials are large enough compared to the coefficients of the linear combination.

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