Derivation of the Wave Equation for Piano Strings and Development of Alternative Tuning Systems for Piano to Minimize Dissonance in Musical Intervals

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The search for a perfect tuning system for music has been a long-standing debate among musical theorists. There is a close relationship between the theory of musical scales and the frequency spectra of musical instruments, which has led to the creation of the present-day tempered scale, with 12 equal divisions of the octave, known as the 12-edo scale. While the 12-edo scale works for many instruments, it is not the most effective solution for achieving consonance in piano music. String instruments like violins and guitars have nearly harmonic frequency spectra, with vibrations following f_n= n*f_0, where n represents the nth partial and f_0 is the fundamental frequency. However, piano tuning is more complex than simply tuning each string to a known fundamental frequency, as the frequency spectrum is inharmonic—slightly higher than the expected frequencies. The following research discusses the derivation of the wave equation for an elastic string and applies the Separation of Variables method to solve a second-order partial differential equation (PDE). The motion of a stiff piano string is modeled by the Euler-Bernoulli beam equation, a fourth order PDE. Solving this equation using the same method as the elastic wave equation results in the formula for the stretched frequency spectrum. The dissonance curve of the frequency spectrum was computed using Python. The local minima of this curve are slightly shifted to the right compared to those of the harmonic spectrum. Therefore, the 12-edo scale does not effectively minimize dissonance, prompting an investigation of various tuning systems based on the alignment of partials in octave and fifth intervals. These systems are analyzed to see which best fit the minima of the dissonance curve near chosen frequency ratios.

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