Prime Numbers of the Form x^2+ny^2 and Thue's Lemma

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The representation of prime numbers as quadratic forms is one of the central topics of number theory, dating back to Fermat and Euler. This study investigates prime numbers expressed as x^2+ny^2 for fixed natural number n and determines the necessary conditions for such representations. Using inequality, quadratic reciprocity, modular arithmetic, basic lemmas and Thue's lemma, we find a new methodology based on applying Thue's lemma to prove that x^2+ny^2 is divisible by a given prime number and then using again Thue's lemma and arithmetic operations we find an upper bound for x^2+ny^2 that is at most n*p. We derive new criteria on p that determine whether every such prime number gives such a remainder, and, using divisibility, modes, and inequalities, check whether this is the only solution for a given n. This methodology shows that for a given n, there are only n possible solutions, and using mod, quadratic reciprocity, and arithmetic operations, we prove that among the remaining choices, the only suitable answer for a prime number is x^2+ny^2=p. This methodology simplifies the solution of the problem for a fixed value of n, and examples have been shown on classical results such as Fermat's two-square theorem, squaring, and two-square multiplication, as well as other cases. These results can make meaningful contributions to education number theory.

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