Novel Metric Operator Theory: A Solution to a Longstanding Open Problem in Operator Theory and Quantum Mechanics

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One of the central open problems in operator theory and PT-symmetric quantum mechanics is how to construct metric operators explicitly for non-Hermitian operators acting in spaces with an indefinite inner product. This problem has been actively studied for more than two decades, beginning with the late-1990s emergence of PT-symmetric quantum mechanics. This project addresses that gap in the finite-dimensional Krein-space setting by developing a constructive theory of metric operators for positive invariant subspaces of J-self-adjoint operators. The main result shows that, whenever a positive invariant subspace is isolated by a spectral gap, the corresponding metric operator is uniquely determined and can be written explicitly as the Gram matrix of the J-inner product. This gives a direct and computable solution to the metric problem in this setting. The work further proves that the metric depends continuously on the subspace and derives a perturbation bound describing how the metric changes under small analytic perturbations. These results turn an abstract existence question into an explicit framework that can actually be calculated. By making the metric operator concrete, this project contributes to the broader open problem of understanding when non-Hermitian operators admit a physically meaningful self-adjoint representation. The theory also supports numerical approximation methods and helps bridge the gap between formal operator theory and applications in quantum mechanics.

Awards Won:
American Mathematical Society: Honorable Mention and One-Year Membership to AMS (for 5 projects with up to 3 team members per project)