Eigenvalue Distributions of Stochastic Matrices and the Karpelevic Region

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Stochastic matrices model finite Markov chains and arise in applications such as search engine algorithms and population dynamics. Their eigenvalues determine long-term behavior, making it a fundamental question which values they can take. The set of all possible eigenvalues of nxn stochastic matrices forms the Karpelevic region, whose boundary was characterized by Karpelevic in 1951 and reformulated by Ito in 1997 using parametric polynomial equations. This study aimed to visualize the Karpelevic region, analyze the eigenvalue distributions of stochastic matrices, and investigate unexpected internal geometric structures that emerged from the computations. For n=4, the boundary was reconstructed by solving the reduced Ito polynomials corresponding to each pair of Farey neighbors and tracing the continuous roots. Random stochastic matrices were generated for n=2-7 by sampling entries from a chi-squared distribution and normalizing each column. Convex combinations between specific pairs of stochastic matrices were constructed to reconstruct internal structures. All computed eigenvalues lay within the unit disk, consistent with the Karpelevic theorem. The Ito polynomials successfully reconstructed the theoretical boundary for n= 4, and all randomly generated eigenvalues lay within this boundary. Pronounced internal arc-like structures were observed, especially for matrices with high column concentration, reproduced through six explicit convex combination constructions. The results confirm the theoretical boundary and reveal an internal geometry of the Karpelevic region depending primarily on the degree of column concentration of the generated matrices. Whether this internal geometry admits a systematic analytic description remains an open question for future investigation.

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