Detecting Causality in (2+1)-Dimensional Globally Hyperbolic Spacetimes Using Conjugation Quandles Over Dihedral Groups

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I investigated whether quandle colorings (homomorphisms) can detect causality for links realized as skies in a (2+1)-dimensional globally hyperbolic spacetime X. I built on the Allen–Swenberg example, where certain 2-sky links were conjectured to be causally related. Allen and Swenberg showed that the Alexander–Conway polynomial does not distinguish these links from the connected sum of two Hopf links H#H, which represents two causally unrelated events. I asked whether quandle invariants can detect the distinction missed by the Alexander–Conway polynomial. I proved that the conjugation quandle of the dihedral group D5 distinguishes the two links: although they share the same Alexander–Conway polynomial, they have different D5 quandle counting invariants. For D5, the counting invariant alone separates the pair, while for other small dihedral groups such as D3, D4, D6, and D7, even the enhanced counting polynomial fails to do so. I further proved that the conjugation quandle over D5n distinguishes the full infinite Allen–Swenberg family from H#H. To explain this, I analyzed reflection and rotation signatures and associated to each coloring an integer matrix determined by the local crossing types. Smith normal form revealed an additional invariant factor pattern for the Allen–Swenberg links, producing a gcd(10,n) term absent for H#H. I showed that the number of quandle homomorphisms satisfies Hom(Q, L) = S a? · n??? · ? gcd(d?, n), which gives a general formula for the number of homomorphisms for all Dn and explains why causality is detected precisely for dihedral groups of the form D5n.

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